Time-Energy Optimal Trajectory Planning over a Fixed Path for a Wheeled Mobile Robot

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Abstract: In this paper a method for time-energy optimal velocity profile planning for a nonholonomic wheeled mobile robot (WMR) is proposed. Instead of relying on a nonlinear programming algorithm, the method utilizes a nonlinear change of variables that can transform the nonlinear optimization problem into a convex optimization problem. The equations are then discretized and later formulated as a second order cone programming that can be solved by the Matlab CVX toolbox. The formulation of the objective function has two components: the total energy and the traversal time that is weighted by a parameter named penalty coefficient. With the use of the penalty coefficient one can easily establish a trade-off between the optimization of total energy and traversal time. If the penalty coefficient is increased then minimization of time is more prioritized than the total energy and vice versa. The formulation gives rise to a Pareto optimality condition from which it is not possible to diminish the traversal time without increasing the total energy and vice versa.

1 INTRODUCTION

The term *trajectory planning* has been used in robotics to refer to the problem of determining both a geometric path and a velocity function for the robot motion. Although these two sub-problems might be solved simultaneously, most often, the problem is solved by first computing a geometric path and then a velocity profile that satisfies a set of constraints (LaValle, 2006)

A geometric path can be parameterised by a single variable representing the robot travels along the path. Once the path is reparameterised in this way, using the robot dynamics, valid velocity profiles can be determined (Bobrow et al., 1985). Therefore, control approaches can be used to find a valid velocity profile that meets certain optimization criteria (Zhao and Tsiotras, 2013; Pham and Stasse, 2015).

Using the Brobow method, Verscheure et al. (Verscheure et al., 2008; Verscheure et al., 2009) present a nonlinear change of variables that transform the time and/or energy optimal trajectory planning problem of a six DOF robotic manipulator into a convex optimal control problem to ensure that an optimal solution is the global one. A Second-Order Cone Programming – SOCP transformation (Lobo et al., 1998) was also used in this approach in order to handle with a linear cost function. Some examples of this approach can be found in the robotics literature (Ardeshiri et al., 2011; Reynoso-Mora et al., 2013; Debrouwere et al., 2013; Zhang et al., 2013).

Lipp and Boyd (Lipp and Boyd, 2014) expand this approach to find the optimal speed profile for minimum traversal time, given the dynamics of holonomic vehicles over a fixed path.

Particularly nonholonomic mobile robots bring an extra challenge for trajectory planning, since they are characterized by kinematic constraints that are not integrable and cannot be eliminated from model equations (Siciliano and Khatib, 2008).

In addition, optimization with non-holonomic models can not be feasible since there are more dynamic equations than optimization variables. Hence, these equations are no longer constraint on the optimization problem.

The objective and the main contribution of this work is to present a both time-energy optimization, expanding from the Lipp and Boyd's approach, for an originally nonholonomic wheeled mobile robot (WMR) model over a fixed path. To this aim, an alternative generalized coordinates system is used to reduce the dynamic equations enough to use them in the optimization, by considering that the fixed path the robot must traverse is itself a constraint.

This paper is organized as follows. In Section 2, a time-energy optimization problem for a class of wheeled mobile robots is formulated. In Section 3, the original problem is transformed into a convex optimization standard form. The problem is discretized in Section 4 in order to solve the problem numerically, and then transformed into a second order cone programming format which is presented in Section 5. Section 6 presents some numerical results and in Section 7 some conclusions about this approach are drawn.

2 PROBLEM STATEMENT

A differential rectangular wheeled mobile robot (WMR) consists of a rigid frame equipped with two independent driven wheels. The robot chassis geometry is defined by the width B and the wheels radii r. Consider $p = [x\ y]^T$ the vector representing the cartesian coordinates of the midpoint P between the wheels.

Usually, the generalized coordinates of the robot are given by the vector $\xi = [x\ y\ \theta]^T$, where θ is the angle between the robot and the inertial coordinate system. This model has a non-holonomic constraint since it is underactuated, i.e., the number of the input signals is lower than the size of the generalized coordinates vector.

Alternatively, consider a vector space where the time parameterised vectors $q : \mathbb{R} \rightarrow \mathbb{R}^2$ are written in the form $q(t) = [\gamma(t)\ \ \theta(t)]^T$, where the monotonically increasing funtion $\gamma : \mathbb{R} \rightarrow \mathbb{R}$ is the distance traversed by the robot on the plane from the time 0 to time the t, and the function $\theta : \mathbb{R} \rightarrow \mathbb{R}$ is the robot angle as a function of the time, as illustrated in Figure 1.

It is easy to prove that the value of the vectors $q(t)$ do not define the position of the robot on the plane but, given a start point p_0, the trajectory of the function q does.

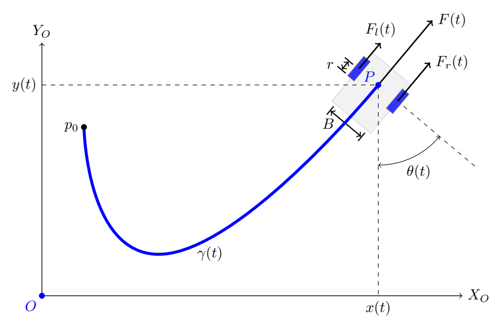

Figure 1: The WMR dimensions and its pose in two different representations ξ and q.

The robot has control inputs $u \in \mathbb{R}^2$, $u = [u_r\quad u_l]^T$, which impose the forces F_r and F_l to the right and left wheels respectively. Given the mass m and the moment of inertia J of the robot structure, consider the simplified robot dynamics written as

$$\mathbf{R}u = \mathbf{M}\ddot{q}, \tag{1}$$

where

$$\mathbf{R} = \begin{bmatrix} \frac{K_m}{r} & \frac{K_m}{r} \\ \frac{K_m B}{2r} & -\frac{K_m B}{2r} \end{bmatrix}, \tag{2}$$

$$\mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}, \tag{3}$$

where K_m is the DC motor torque constant, and $\ddot{q} \in \mathbb{R}^2$, $\ddot{q} = [\ddot{\gamma}\quad \ddot{\theta}]^T$ is the acceleration vector.

2.1 Geometric Path Time Parameterisation

Consider a *geometric path* as a function $s : [0,1] \rightarrow \mathbb{R}^2$ that describes the WMR motion, such that

$$s(\tau(t)) = q(t), \quad t \in [0, T_f]. \tag{4}$$

The function $\tau : [0, T_f] \rightarrow [0,1]$, named *virtual time*, maps the robot motion time in a normalized interval where $\tau(0) = 0$ e $\tau(T_f) = 1$. This function is defined as monotonically increasing, i.e., $\dot{\tau}(t) > 0$, along the interval $\tau = [0,1]$. The derivatives of Equation 4 are the robot velocity and acceleration in the configuration space:

$$\dot{q}(t) = s'(\tau)\dot{\tau}(t) \tag{5}$$

$$\ddot{q}(t) = s'(\tau)\ddot{\tau}(t) + s''(\tau)\dot{\tau}^2(t), \tag{6}$$

where $(.)'$ refers to the derivatives in relation with τ.

2.2 Initial problem definition

Consider a differential WMR with initial position $p(0) = p_0$ and initial configuration $q(0) = q_0$ moving along a pre-defined path $s(\tau(t))$ in a time interval $\{t \in \mathbb{R} \mid t = [0, T_f]\}$ up to reach a final configuration $q(T_f) = q_f$.

We want to find a velocity profile $\dot{q}(t)$, $t \in [0, T_f]$, such that the robot is driven through the path s in the minimum traversal time T_f and energy consumption E_t. This problem can be defined as an optimization problem that can be described as:

Problem 1. (Minimal time-energy problem)

$$\underset{(u,\tau)}{min.}: \quad \mathcal{J}(t,u) = \int_0^{T_f} \|u(t)\|_2^2 \, dt + \mu \int_0^{T_f} dt \tag{7}$$

$$s.t.: \quad \mathbf{R}u(t) = \mathbf{M}\ddot{q}(t) \tag{8}$$

$$q(t) = s(\tau(t)) \tag{9}$$

$$u_{min} \leq u(t) \leq u_{max} \tag{10}$$

$$\dot{q}_{min} \leq \dot{q}(t) \leq \dot{q}_{max} \tag{11}$$

$$\ddot{q}_{min} \leq \ddot{q}(t) \leq \ddot{q}_{max}, \ t \in [0, T_f] \tag{12}$$

The optimization variables are the virtual time τ and the input vector u, while the matrices $\mathbf{R}$, $\mathbf{M}$, the path s and the penalty coefficient $\mu \in \mathbb{R}_+$ are given data. The notations $(.)_{min}$ and $(.)_{max}$ refer respectively to the minimum and maximum values, and the inequalities given by Equations 10–12 define a polyhedra $\mathcal{C}_t \subseteq \mathbb{R}^{2x2x2}$ such that $(u(t), \dot{q}^2(t), \ddot{q}(t)) \in \mathcal{C}_t$.

3 CONVEXIFICATION

Using Equation 6 in Equation 1 that defines the WMR dynamics, yields (Verscheure et al., 2008; Lipp and Boyd, 2014):

$$\mathbf{R}u(\tau) = \mathbf{M}\left(s'(\tau)\ddot{\tau}(t) + s''(\tau)\dot{\tau}^2(t)\right). \tag{13}$$

Two new functions are introduced,

$$a(\tau) = \ddot{\tau}, \tag{14}$$

$$b(\tau) = \dot{\tau}^2. \tag{15}$$

One can note that

$$\dot{b}(\tau) = b'(\tau)\dot{\tau}, \tag{16}$$

$$\dot{b}(\tau) = \frac{d(\dot{\tau}^2)}{dt} = 2\ddot{\tau}\dot{\tau} = 2a(\tau)\dot{\tau}, \tag{17}$$

which results in the following relation

$$b'(\tau) = 2a(\tau). \tag{18}$$

The objective function given by Equation 7 can be redefined, using Equation 15, as

$$\mathcal{J}(\tau,u) = \int_{\tau(0)}^{\tau(T_f)} [\|u(\tau)\|_2^2 + \mu] \frac{d\tau}{\dot{\tau}} = \\ = \int_0^1 \frac{[\|u(\tau)\|_2^2 + \mu]}{\sqrt{b(\tau)}} \, d\tau, \tag{19}$$

and the nonlinear Problem 1 becomes equivalent to the following problem:

Problem 2. (Minimal Time-Energy Convex Problem)

$$\underset{(u,a,b)}{min.}: \mathcal{J}(\tau,u) = \int_0^1 \frac{[\|u(\tau)\|_2^2 + \mu]}{\sqrt{b(\tau)}} \, d\tau \tag{20}$$

$$s.t.: \mathbf{R}u(\tau) = \mathbf{M}\left(s'(\tau)a(\tau) + s''(\tau)b(\tau)\right), \tag{21}$$

$$b'(\tau) = 2a(\tau), \tag{22}$$

$$(u(\tau), a(\tau), b(\tau)) \in \mathcal{C}_\tau, \ \tau \in [0,1], \tag{23}$$

where

$$\mathcal{C}_\tau = \{(u(\tau), a(\tau), b(\tau))| \\ (u(\tau), s'^2(\tau)b(\tau), s'(\tau)a(\tau) + s''(\tau)b(\tau)) \in \mathcal{C}_t\}.$$

The Problem 2 is a convex optimization problem, since the objective function (Equation 20) is convex, the equality constraints (Equations 21 and 22) are afine function of the optimization variables and $\mathcal{C}_\tau$ is a convex set (Boyd and Vandenberghe, 2004).

4 DISCRETE LINEARIZATION

The virtual time $\tau(t)$ is an independent variable in Problem 2. In order to solve the problem numerically it is necessary to reformulate Equations 20–23 from continuous time to discrete time. In this context, the virtual time τ is discretized into $N+1$ points such that $\tau_0 = 0 \leq \tau_i \leq 1 = \tau_N, i = 1, \ldots, N-1$.

On the same way, variables $u(\tau)$, $a(\tau)$ and $b(\tau)$ have to be discretized into a finite number of variables: $u_i \in \mathbb{R}^2$; $a_i \in \mathbb{R}$, $b_i \in \mathbb{R}$; $i = 1, \ldots, N$; and path $s(\tau)$ and its derivatives.

4.1 Discretization of Optimization Variables

If the time interval between two consecutive points is small enough, it is possible to consider that variables $a(\tau)$ and $u(\tau)$ are constant between two consecutive time instants τ_{i-1} and τ_i, $i = 1, \ldots, N$. Therefore, both $a(\tau)$ and $u(\tau)$ are discontinuous in virtual time τ. These variables are evaluated in the middle point between two consecutive points as it follows, for all $i = 1, \ldots, N$,

$$\bar{\tau}_i = \frac{\tau_i + \tau_{i-1}}{2}, \tag{24}$$

and

$$a_i = a(\bar{\tau}_i) = a\left(\frac{\tau_i + \tau_{i-1}}{2}\right), \tag{25}$$

(See Figure 2) and,

$$u_i = u(\bar{\tau}_i) = u\left(\frac{\tau_i + \tau_{i-1}}{2}\right). \tag{26}$$

On the other hand, b is evaluated in each discrete point $b_i = b(\tau_i)$, $i = 1,\ldots,N$. $b_0 = b(\tau_0)$ is defined as a parameter of the problem.

Since it is assumed that $a(\tau)$ is constant and $b(\tau)$ is affine across two consecutive discrete points, $b(\tau)$ might be estimated by a first-order *Taylor* series expansion:

$$b(\tau) = b_{i-1} + \left(\frac{b_i - b_{i-1}}{\tau_i - \tau_{i-1}}\right)(\tau - \tau_{i-1}), \tag{27}$$

for $i = 1,\ldots,N$, (See Figure 3), while for the middle point $\bar{\tau}_i$ the estimate is given by:

$$b(\bar{\tau}_i) = \frac{b_i + b_{i-1}}{2}, \tag{28}$$

for all $i = 1,\ldots,N$.

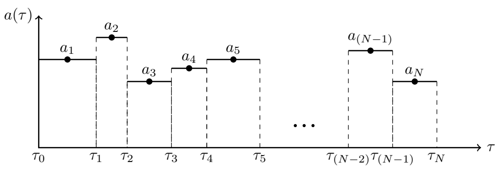

Figure 2: $a(\tau)$ is piecewise constant for $\tau \in [0,1]$ (Adapted from (Verscheure et al., 2008)).

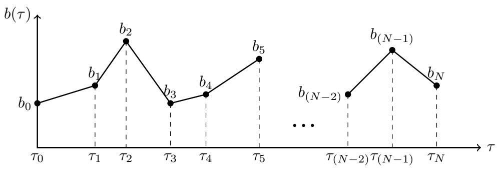

Figure 3: $b(\tau)$ is piecewise linear for $\tau \in [0,1]$ (Adapted from (Verscheure et al., 2008)).

4.2 Discretization of the Optimization Equations

The objective function of Problem 2 given by Equation 20 might be rewritten as a sum of definite integrals:

$$\mathcal{J}(\tau,u) = \sum_{i=1}^{N} \int_{\tau_{i-1}}^{\tau_i} \frac{[\|u(\bar{\tau})\|_2^2 + \mu]}{\sqrt{b(\tau)}}\, d\tau. \tag{29}$$

Since $u(\tau)$ is constant in each interval it is possible to write:

$$\mathcal{J}(\tau,u) = \sum_{i=1}^{N} \left\{ (\|u_i(\bar{\tau})\|_2^2 + \mu) \int_{\tau_{i-1}}^{\tau_i} \frac{1}{\sqrt{b(\tau)}}\, d\tau \right\}. \tag{30}$$

Using Equation 27 and after some algebraic transformations,

$$\mathcal{J}(\tau,u) = \sum_{i=1}^{N} \frac{2(\|u_i\|_2^2 + \mu)(\tau_i - \tau_{i-1})}{\left(b_i^{1/2} + b_{i-1}^{1/2}\right)}. \tag{31}$$

The continuous dynamics described by Equation 21 evaluated in the virtual instant time $\bar{\tau}_i$ is given by:

$$\mathbf{R}u(\bar{\tau}_i) = \mathbf{M}\left[s'(\bar{\tau}_i)a(\bar{\tau}_i) + s''(\bar{\tau}_i)b(\bar{\tau}_i)\right] \tag{32}$$

In order to discretize Equation 32 derivatives of path s must be estimated. If path s is given in a discrete way, $s_i \in \mathbb{R}^3$, $s_i = s(\tau_i)$, $i = 0,\ldots,N$ the value of s in the middle point between two consecutive discrete points might be evaluated as

$$\bar{s}_i = s(\bar{\tau}_i) = \frac{s_i + s_{i-1}}{2}, \tag{33}$$

and the correponding first derivative as

$$\bar{s}'_i = s'(\bar{\tau}_i) = \frac{s_i - s_{i-1}}{\tau_i - \tau_{i-1}}, \tag{34}$$

$i = 1,\ldots,N$.

For the second derivative, an approximation based on the sixth-order *Runge-Kutta* symmetric method is utilized (Lipp and Boyd, 2014):

$$\bar{s}''_i = s''(\bar{\tau}_i) = \frac{-\frac{5}{48}s_{i-3} + \frac{13}{16}s_{i-2} - \frac{17}{24}s_{i-1}}{(\tau_i - \tau_{i-1})^2} \times \\ \times \frac{-\frac{17}{24}s_i + \frac{13}{16}s_{i+1} - \frac{5}{48}s_{i+2}}{(\tau_i - \tau_{i-1})^2}. \tag{35}$$

Since $i = 1\ldots N$, Equation 35 can not be evaluated at time instants $\bar{\tau}_1, \bar{\tau}_2, \bar{\tau}_{N-1}$ and $\bar{\tau}_N$. Assuming that the path $s(\tau)$ is linear at its start and at its end, the second derivatives close to the extremal points might be evaluated using the fourth-order *Runge-Kutta* symmetric method as

$$\bar{s}''_i = \frac{s_{i-2} - s_{i-1} - s_i + s_{i+1}}{2(\tau_i - \tau_{i-1})^2}. \tag{36}$$

Using this scheme, estimates $s_{-1} = 2s_0 - s_1$ and $s_{N+1} = 2s_N - s_{N-1}$ are obtained.

The complete algorithm is defined recursively as

$$\bar{s}''_1 = \frac{s_0 - 2s_1 + s_2}{2(\tau_1 - \tau_0)^2}, \tag{37}$$

$$\bar{s}''_2 = \frac{s_0 - s_1 - s_2 + s_3}{2(\tau_2 - \tau_1)^2}, \tag{38}$$

$$\bar{s}''_{(N-1)} = \frac{s_{(N-3)} - s_{(N-2)} - s_{(N-1)} + s_{(N)}}{2(\tau_{(N-1)} - \tau_{(N-2)})^2}, \tag{39}$$

$$\bar{s}''_N = \frac{s_{(N-2)} - 2s_{(N-1)} + s_N}{2(\tau_N - \tau_{(N-1)})^2}. \tag{40}$$

Finally, starting from Equations 31 and 32, Problem 2 is written in discrete-time as:

Problem 3. (Discrete Minimum Time-Energy Convex Problem)

$$\underset{(u_i,a_i,b_i)}{min.} : \quad J(\tau,u) = \sum_{i=1}^{N} \frac{2(\|u_i\|_2^2+\mu)}{\left(\sqrt{b_i}+\sqrt{b_{i-1}}\ \right)}(\delta\tau_i) \tag{41}$$

$$s.t. : \quad \mathbf{R}u_i = \mathbf{M}\left[\bar{s}_i' a_i + \frac{\bar{s}_i''}{2} b_i + \frac{\bar{s}_i''}{2} b_{i-1}\right], \tag{42}$$

$$b_i - b_{i-1} = 2a_i(\delta\tau_i) \tag{43}$$

$$(u_i,a_i,b_i) \in \mathcal{C}_{\bar{\tau}},\ i=1,\ldots,N, \tag{44}$$

where

$$\mathcal{C}_{\bar{\tau}} = \{(u_i,a_i,b_i)| \\ (u(\bar{\tau}), s'^2(\bar{\tau})b(\bar{\tau}), s'(\bar{\tau})a(\bar{\tau}) + s''(\bar{\tau})b(\bar{\tau})) \in \mathcal{C}_t\},$$

and $\delta\tau_i = \tau_i - \tau_{i-1}$.

5 FORMULATION AS A SOCP

Though the Problem 3 is a discrete convex optimization problem, the objective function in the Equation 41 is nonlinear. Therefore, four new variables $c_i, d_i, e_i, f_i \in \mathbb{R}$ are introduced in Problem 3, where $c_i > 0$, $d_i > 0$, $e_i > 0$, $f_i > 0$, and $i = 1,\ldots,N$, in order to redefine the objective function as

$$\begin{aligned}\tilde{\mathcal{J}}(\tau,u) &= \sum_{i=1}^{N} 2(e_i+\mu f_i)\delta\tau_i \geq \\ &\geq \sum_{i=1}^{N} \frac{2(\|u_i\|_2^2+\mu)}{\left(\sqrt{b_i}+\sqrt{b_{i-1}}\ \right)}(\delta\tau_i) = \\ &= \mathcal{J}(\tau,u),\end{aligned} \tag{45}$$

where the inequalities constraints

$$\frac{1}{d_i} \leq f_i, \tag{46}$$

$$\frac{u_i^T u_i}{d_i} \leq e_i, \tag{47}$$

$$d_i \leq c_i + c_{i-1}, \tag{48}$$

$$c_i \leq \sqrt{b_i}, \tag{49}$$

can be considered as constraints which minimize the objective function, i.e., $\tilde{\mathcal{J}} \geq \mathcal{J}$ is an upper constraint bound of $\mathcal{J}$, it means that minimizing $\tilde{\mathcal{J}}$ ensures that $\mathcal{J}$ is minimal. Moreover, $\tilde{\mathcal{J}}$ is a linear function of the new optimization variables.

Theorem 1 (Hyperbolic constraints as SOC constraints)**.** *Given variables $w,x,y \in \mathbb{R}$ with $x \geq 0$, $y \geq 0$ then (Lobo et al., 1998):*

$$w^2 \leq xy \iff \left\| \begin{bmatrix} 2w \\ x-y \end{bmatrix} \right\|_2 \leq x+y,$$

and, more generically, when $w \in \mathbb{R}^n$,

$$w^T w \leq xy \iff \left\| \begin{bmatrix} 2w \\ x-y \end{bmatrix} \right\|_2 \leq x+y.$$

Applying Theorem 1 in the inequalities 46, 47 and 49 it follows

$$\left\| \begin{bmatrix} 2 \\ d_i - f_i \end{bmatrix} \right\|_2 \leq d_i + f_i \tag{50}$$

$$\left\| \begin{bmatrix} 2u_i \\ e_i - d_i \end{bmatrix} \right\|_2 \leq e_i + d_i \tag{51}$$

$$\left\| \begin{bmatrix} 2c_i \\ b_i - 1 \end{bmatrix} \right\|_2 \leq b_i + 1 \tag{52}$$

Hence, Problem 3 is equivalent to:

Problem 4. (Discrete Minimal Time-Energy Convex Problem as a SOCP)

$$\underset{\binom{u_i}{a_i,\cdots,f_i}}{min} : \quad \mathcal{J}(\tau,u) = \sum_{i=1}^{N} 2(e_i+\mu f_i)\delta\tau_i \tag{53}$$

$$s.t. : \quad \left\| \begin{bmatrix} 2 \\ d_i - f_i \end{bmatrix} \right\|_2 \leq d_i + f_i \tag{54}$$

$$\left\| \begin{bmatrix} 2u_i \\ e_i - d_i \end{bmatrix} \right\|_2 \leq e_i + d_i \tag{55}$$

$$\left\| \begin{bmatrix} 2c_i \\ b_i - 1 \end{bmatrix} \right\|_2 \leq b_i + 1 \tag{56}$$

$$d_i \leq c_i + c_{i-1} \tag{57}$$

$$\mathbf{R}u_i = \mathbf{M}\left[\bar{s}_i' a_i + \frac{\bar{s}_i''}{2} b_i + \frac{\bar{s}_i''}{2} b_{i-1}\right], \tag{58}$$

$$b_i - b_{i-1} = 2a_i(\delta\tau_i) \tag{59}$$

$$(u_i,a_i,b_i) \in \mathcal{C}_{\bar{\tau}},\ i=1,\ldots,N, \tag{60}$$

Solving the Problem 4, the optimal input signal u_i^* and the optimal auxiliary variables $(a^*,\cdots,f^*)$ are found. Thereby, the time t can be re-parametrized by the rate

$$\delta t_i = \frac{\delta\tau_i}{\sqrt{b^*(\tau_i)}}, \tag{61}$$

as well as the traversal time in seconds and the total energy consumption in V^2 are calculated by the sums

$$T_f = \sum_{1}^{N} \frac{\delta\tau_i}{\sqrt{b_i^*}}, \tag{62}$$

$$E_t = \sum_{1}^{N} \|u_i^*\|_2^2. \tag{63}$$

6 RESULTS

In order to validate the proposed method some experimental results are explored in this section.

A smoothing spline p is defined with smoothing parameter $\rho = 0.99$ to reach the following waypoints : $p_0 = [0 \quad 0]^T$, $wp_1 = [2 \quad -1]^T$, $wp_2 = [3 \quad -1]^T$, $wp_3 = [6 \quad -1]^T$, $wp_4 = [8 \quad -1]^T$ and $wp_5 = [5 \quad -1]^T$.

The spline is then divided into $N = 500$ parts of equal length, resulting in $N+1$ discrete points $p_0, \ldots, p_N$. Figure 4 illustrates the smoothing spline p.

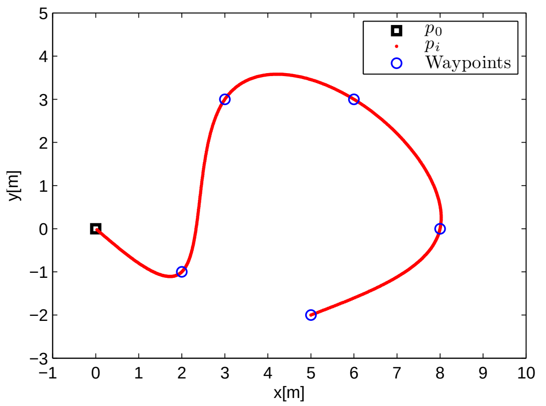

Figure 4: Spline p in the Cartesian plane. The black squared mark represents the initial point p_0 and the blue circles represent the *waypoints* wp_i, $i = 1, \ldots, 5$. The red dots are the discrete points p_i, $i = 1, \ldots, N$.

Each point $p_i = [x_i \quad y_i]^T$ is described by the alternative description in the configuration space $s_i = [\gamma_i \; \theta_i]^T$.

The middle points $\bar{s}_i, (i = 1, \ldots, N)$ are calculated by Equation 33, the corresponding estimates of the 1st order derivative $\bar{s}'_i$ by Equation 34 and the 2nd order derivative $\bar{s}''_i$ by Equation 35 and Equations 36 for the extremal points of the curve.

A wheeled robot with the following characteristics are considered: width $B = 0.4m$, inertial mass $m = 10kg$, moment of inertia $J = 2.833Kg/m^2$, wheel radius $r = 0.1m$, DC motor torque constant $K_m = 65 \cdot 10^{-3} Nm/V$, nominal input voltage $u_{max} = -u_{min} = 12V$.

The robot motion has the following constraints maximum velocity $\dot{q}_{max} = [2.5m/s \quad 1rad/s]^T$ and maximum acceleration $\ddot{q}_{max} = [2m/s^2 \quad .5rad/s^2]^T$. Also the initial value for $b_0 = b(\tau_0) = 0$. No final velocity constraint is imposed.

Problem 4 is solved using the algorithm Mosek of the Matlab CVX toolbox (Grant and Boyd, 2014).

The optimization variables are the left and right motors input voltage $u_i = [u_{ri} \quad u_{li}]^T$, auxiliary variables a_i and b_i, and also auxiliary variables c_i, d_i, e_i e f_i, that are defined on Equations 46–49, $i = 1, \ldots, N$. A desktop computer equipped with a 3.60Ghz clock Intel Core i7-4790, 8GB of RAM running Windows 8 x64 is utilized.

Optimization trials have been performed while varying the penalty coefficient from $\mu = 10^{-4}$ up to $\mu = 10^4$, with a step size $\delta\mu = 10^{2.5}$. Based on these trials a graph is designed that represents the traversal time versus total energy $T_f \times E_t$ parametrized by the penalty coefficient μ (See Figure 5).

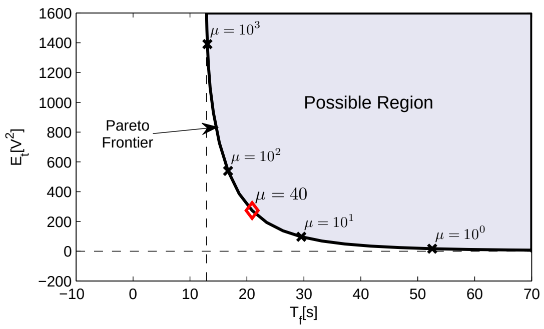

Figure 5: Graph of $T_f \times E_t$. The shaded region is a possible set of $T_f \times E_t$, limited by the Pareto Front, which converges asymptotically to the dotted lines. Some values of μ related to powers of 10 are represented by x marks. $\mu = 40$ is emphasized by a red squared mark.

When the penalty coefficient μ decreases towards zero, $\mu \to 0$, the optimization algorithm gradually diminishes the importance of the traversal time T_f while increasing the importance of the total energy E_t. As a consequence the robot velocity diminishes thus increasing the traversal time T_f. The extreme situation of $\mu = 0$,i.e., only optimization of the total energy E_t is of concern, the traversal time is equivalent to $T_f = 1218s$ while consuming $E_t = 1.4mV^2$.

On the contrary, if only the optimization of the traversal time T_f is important than one can set a very large penalty coefficient $\mu \to \infty$. In this case, the traversal time is saturated in $T_f = 12.8s$ and the energy effort saturated in $E_t = 1704V^2$.

Figure 6 illustrates the series of the optimal input voltage signals (a) u_r^* and (b) u_l^* and the optimal velocity profile (c) v^* with variation of the coefficient μ. The arrows shows that the higher the value of μ, the greater the effort of the motor; consequently, the greater the linear velocity over the path.

Note some saturation in the input signals (e.g. on u_r^* at $i = 250$ and on u_l^* at $i = 120$) and the linear velocity saturation $v_{max} = 2.5m/s$ on the inverval $[285 \leq i \leq 332]$.

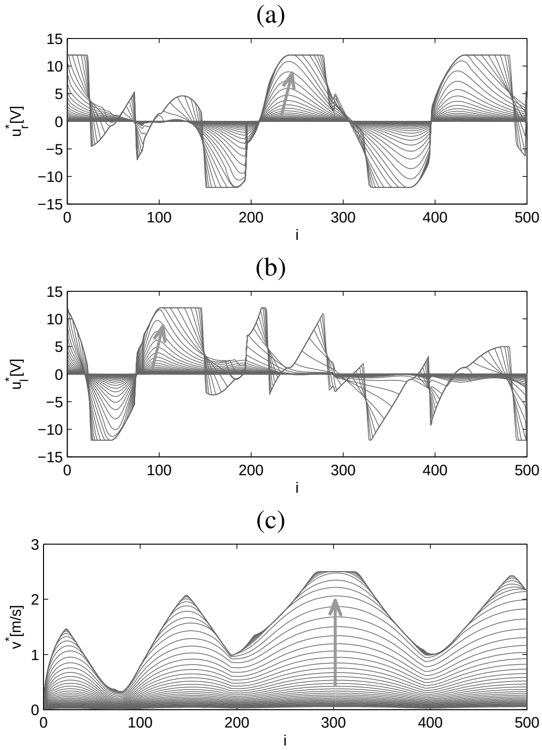

Figure 6: Optimal actuators input voltage signals (a) u_r^*, (b) u_l^* and (c) velocity profile v^* in function of i. The arrows indicate the signal trend with the increasing of the penalty parameter μ.

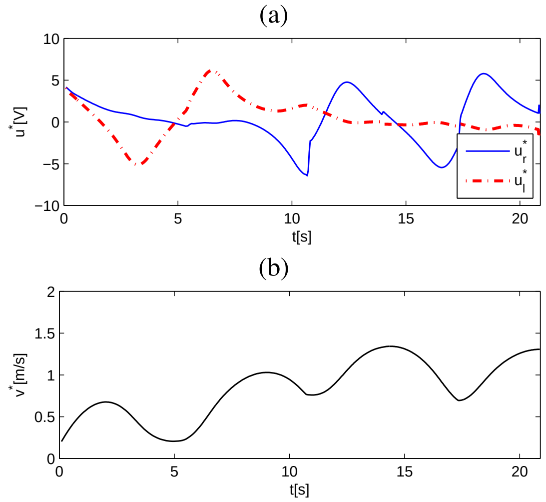

Figure 7: Optimal (a) actuators input voltage signals and (b) corresponding velocity profile for $\mu = 40$.

In Figure 5, it is possible to note a Pareto Knee Point from which it is not possible to diminish the traversal time T_f without increasing the total energy E_f and vice versa. The red squared mark emphasizes this point which corresponds to a penalty coefficient $\mu = 40$. The Figure 7 illustrates the optimal signals for this particular case, where the traversal time is $T_f = 20.9s$ and total energy $E_t = 274V^2$.

7 CONCLUSIONS

In this paper a method for time-energy optimal velocity profile planning for a nonholonomic wheeled mobile robot is proposed.

The main contribution of this work is to consider an alternative generalized configuration that allows the use of the previous optimization method (Verscheure et al., 2008; Lipp and Boyd, 2014) for a class of non-holonomic WMR.

The optimization problem is discretized and later formulated as a second order cone programming that can be solved by the algorithm Mosek of the Matlab CVX toolbox.

The formulation of the objective function has two components: the total energy E_t and the traversal time T_f that is weighted by a parameter named penalty coefficient μ.

As the total energy E_t is directly related with the vehicle speed which is inversely related to the traversal time T_f, only one penalty coefficient μ is necessary to complete specify the trade-off between the total energy and the traversal time. In this context the format of the graph of traversal time versus total energy $T_f \times E_t$ parametrized by the penalty coefficient μ (see Figure 5) comes as no surprise.

It is possible to note that the prioritization of the traversal time is limited by hard constraints like limitation on actuators input voltage, maximum angular velocity and maximum angular acceleration.

There is a Pareto Knee point from which it is not possible to diminish the traversal time T_f without increasing the total energy E_f and vice versa. A coefficient μ auto tunning method to find this Pareto Knee is considered by the authors as a future work.

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